An Eddy Closure for Potential Vorticity

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ABSTRACT

The Gent-McWilliams (GM) parameterization is extended to include a direct influence in the momentum equation. The extension is carried out in two stages; an analysis of the inviscid system is followed by an analysis of the viscous system. In the inviscid analysis the momentum equation is modified such that potential vorticity is conserved along particle trajectories following a transport velocity that includes the Bolus velocity in a manner exactly analogous to the continuity and tracer equations. In addition (and in contrast to traditional GM closures), the new formulation of the inviscid momentum equation results in a conservative exchange between potential and kinetic energy. The inviscid form of the eddy closure does not conserve total energy; there is a non-conservation proportional to the time derivative of the Bolus velocity. The hypothesis that the viscous term in the momentum equation should give rise to potential vorticity being diffused along isopycnals in a manner analogous to other tracers is examined in detail. While the form of the momentum closure that follows from a strict adherence to this hypothesis is not immediately interpretable within the constructs of traditional momentum closures, three approximations to this hypothesis results in a form of dissipation that is consistent with traditional Laplacian diffusion. The most important conclusion from this hypothesis is that the horizontal viscosity parameter in the momentum equation should be identical to the traditional GM closure parameter. We propose the viscous form of the eddy closure for potential vorticity as a possible closure for use in ocean circulation models.

1. Introduction

The standard implementation of the Gent and McWilliams (1990) closure for the effects of mesoscale eddies on the mean flow is in the equations for potential temperature and salinity in z-coordinates, or in the layer thickness and tracer equations in isopycnal coordinates. In virtually all ocean models and ocean components of climate models, the momentum equation used is the primitive equation form for the mean velocity.

For a constant coefficient, the Gent and McWilliams (1990) closure assumes a down-gradient assumption for layer thickness variations in isopycnal coordinates to represent the additional Bolus velocity acting on tracers. There have been many suggestions, both from theoretical ideas and analyses of numerical model results, that the Bolus velocity should be based on a down-gradient assumption for potential vorticity (PV). However, the model analyses always use the approximate planetary potential vorticity, the Coriolis parameter divided by the layer thickness, which only differs from the GM form by the Coriolis parameter. This is not the true PV of the primitive equations, which is expressed as the absolute vorticity divided by the isopycnal layer thickness. In this paper, the consequences of an eddy parameterization for the full Ertel PV, based on the absolute vorticity, is explored in detail. A consequence is that the eddy parameterization also affects the vorticity equation, and hence the momentum equation of the model.

There have been several previous proposals to use different momentum equations in non-eddy-resolving models, although none have been implemented in standard, global ocean models. Gent and McWilliams (1996) suggest that momentum advection should be by the transport velocity, not the mean velocity, to be consistent with the tracer advection.

Greatbatch (1998) and McDougall and McIntosh (1996) both propose a more radical change; namely that the momentum equation should be written entirely in terms of the transport velocity. One reason is that the entire model then has only a single velocity variable. A numerical model using a momentum equation of this form has been implemented, and used to obtain global solutions, by Ferreira and Marshall (2006). However, these solutions differ near the equator from those using the standard GM form, because the geostrophic approximation is used to transform GM into a vertical viscosity term, see Greatbatch (1998) and Gent et al. (1995). With this form of the momentum equation, a mean PV is conserved, but it is a function of the transport velocity, which is difficult to justify for the following reason.

Ertel PV is often assumed to be the most fundamental dynamical variable because it obeys a conservation equation, and all the other variables can be determined if the PV distribution is known. PV satisfies the same inviscid, adiabatic equation as a passive tracer; so it is often assumed that it should be treated exactly like a passive tracer. In isopycnal coordinates, this means that the averaging should be done on the equation for the thickness times the PV, which is just the absolute vorticity. This is a linear function of velocity, so that averaging results in the mean PV being a function of the mean velocity, not the transport velocity. In this paper, we assume the hypothesis that mean Ertel PV should obey the same conservation equation as a passive tracer, and is a function of the mean velocity. This requires that the momentum equation solves for the mean velocity, not the transport velocity. We propose a different momentum equation that results in Ertel PV being conserved along particle trajectories defined by the transport velocity, just like a passive tracer. An adiabatic analysis of the PV and energy equations is given in Sections 2 and 3, and Section 4 explores the consequences of assuming that PV is also diffused along isopycnal surfaces.

2. Analysis of potential vorticity

a. The unmodified continuous system

The analysis is based on incompressible, Bousinesq and adiabatic flow in isopycnal coordinates. The isopycnal layer thickness equation can be expressed as

$$\frac{\partial \mu}{\partial t} + \nabla_\rho \cdot (\mu \mathbf{u}) = 0, \tag{1}$$

where the isopycnal layer thickness, μ, is defined as $\mu = \partial h / \partial \rho$ and h is the height of constant density surfaces. The layer thickness is transported by the horizontal velocity $\mathbf{u}$. The entire analysis is conducted in an isopycnal coordinate system where the ∇, $\nabla\cdot$ and $\mathbf{k}\cdot\nabla\times$ operators are always evaluated along isopycnal surfaces, hence we omit the ρ subscript on these operators below. The inviscid momentum equation can be expressed as

$$\frac{\partial \mathbf{u}}{\partial t} + (\zeta + f)\, \mathbf{k} \times \mathbf{u} = -\nabla\phi - \nabla K, \tag{2}$$

where ζ is the relative vorticity defined as $\zeta = \mathbf{k} \cdot \nabla \times \mathbf{u}$, $\mathbf{k}$ is the unit vector defining the local vertical direction, f is the planetary vorticity projected in the direction of the local vertical, ϕ is the Montgomery potential and K is the kinetic energy defined as $\frac{1}{2}\,|\mathbf{u}\cdot\mathbf{u}|$. The absolute vorticity equation is obtained by applying the $\mathbf{k}\cdot\nabla\times$ operator to (2) to obtain

$$\frac{\partial \eta}{\partial t} + \nabla \cdot (\eta \mathbf{u}) = 0, \tag{3}$$

where the absolute vorticity is defined as $\eta = \mathbf{k} \cdot \nabla \times \mathbf{u} + f$. In the isopyncal system, Ertel's potential vorticity is defined as $q = \eta/\mu$ and can be used to rewrite (3) as a potential vorticity equation of the form

$$\frac{\partial}{\partial t}(\mu\, q) + \nabla \cdot (\mu\, q\, \mathbf{u}) = 0. \tag{4}$$

When (4) and (1) are combined, the PV equation can be expressed as

$$\frac{Dq}{Dt} = \frac{\partial q}{\partial t} + \mathbf{u} \cdot \nabla q = 0, \tag{5}$$

which states that in the inviscid system PV is a conserved along particle trajectories following the $\mathbf{u}$ velocity. Equation (4) indicates that the time-rate-of-change of density-weighted PV at a fixed position is due entirely to the divergence of the density-weighted PV flux. If we track the $\nabla \cdot (\mu\, q\mathbf{u})$ term in (4) backwards in the derivation to its root, we find it arises *entirely* from the second term in the momentum equation (2). We refer to the $(\zeta + f)\,\mathbf{k} \times \mathbf{u}$ in (2) as the nonlinear Coriolis force since it includes the quasi-linear Coriolis force due to planetary vorticity and a portion of the nonlinear velocity transport term. It is critically important to recognize that

$$\mathbf{k} \cdot \nabla \times [(\mu\, q)\,\mathbf{k} \times \mathbf{u}] = \nabla \cdot (\mu\, q\mathbf{u})\,. \tag{6}$$

The divergence of PV flux shown in (4) is *exactly* equal to the curl of the nonlinear Coriolis force shown in (2). Furthermore, PV is transported by the velocity that appears in the nonlinear Coriolis force.

b. Defining an inviscid eddy closure on PV

The underlying premise of the GM closure is that when certain processes are missing in a simulation, such as meso-scale eddies in coarse-resolution ocean simulations, then the velocity used to transport layer thickness and tracer constituents differs from the predicted, mean velocity if it is to include the effect of these unresolved processes (Gent and McWilliams 1990). Formally, we express the differences between these two types of velocity as

$$\mathbf{U} = \mathbf{u} + \mathbf{u}^*, \tag{7}$$

where the mean velocity $\mathbf{u}$ is predicted from the momentum equation, $\mathbf{U}$ is the velocity used to transport layer thickness and tracer constituents and $\mathbf{u}^*$ is the difference between these two velocities. In the context of the GM closure, $\mathbf{u}^*$ is commonly referred to as the Bolus velocity (Gent et al. 1995). The Bolus velocity is computed based on state variables and accounts for, at least partially, the influence of unresolved phenomena. The precise form of the closure is not required for the analysis below, i.e. the analysis holds for any type of closure that results in the transport velocity differing from the mean velocity. In typical implementations of the GM closure, the isopycnal layer thickness equation is altered to become

$$\frac{\partial \mu}{\partial t} + \nabla \cdot (\mu \mathbf{U}) = 0, \tag{8}$$

and all tracer constituent equations are similarly changed to

$$\frac{\partial}{\partial t}(\mu\,\tau) + \nabla \cdot (\mu\,\tau\,\mathbf{U}) = 0. \tag{9}$$

In effect, the material derivative is altered to become

$$\frac{D^*}{Dt} = \frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla, \tag{10}$$

which states that Lagrangian conserved quantities are transported by the **U** velocity.

While the GM closure impacts the evolution of isopycnal layer thickness and tracer constituents, in general the momentum equation is unaltered, i.e. the GM closure does not directly modify the mean velocity field **u** and (2) is used unchanged even when the GM closure is incorporated into the simulation. As discussed in the Introduction, our hypothesis is that the eddy closure gives a mean PV that obeys a conservation equation like (5). However, if we try to repeat the derivation of the PV equation shown in Section 2.a based on (8) and (2), we find that we can not obtain a PV equation of the form shown in (5). The root cause for this difficulty is that the former equation is based on the material derivative shown in (10) while the latter is based on a material derivative shown in (5). A PV relation analogous to (5) requires that these material derivatives be the same.

When the momentum equation is expressed in vector-invariant form, as shown in (2), and the relationship between the nonlinear Coriolis force and the PV flux is recognized, as shown in (6), then understanding how to modify the momentum equation to obtain the assumed PV dynamics is straightforward. The PV flux in (4) is based on the **u** velocity that is identical to, and arises completely from, the **u** velocity in the nonlinear Coriolis force. Thus, the appropriate modification of (2) is simply

$$\frac{\partial \mathbf{u}}{\partial t} + (\zeta + f)\,\mathbf{k} \times \mathbf{U} = -\nabla\phi - \nabla K. \tag{11}$$

Equation (11) states that the inviscid part of the GM closure on momentum can be captured by computing the nonlinear Coriolis force based on the transport velocity instead of the predicted, mean velocity. If we repeat the derivation of PV in Section 2.a based on (11), we obtain

$$\frac{\partial}{\partial t}(\mu\, q) + \nabla \cdot (\mu\, q\, \mathbf{U}) = 0, \tag{12}$$

that when combined with (8) leads to

$$\frac{D^* q}{Dt} = \frac{\partial q}{\partial t} + \mathbf{U} \cdot \nabla q = 0. \tag{13}$$

Computing the nonlinear Coriolis force based on $\mathbf{U}$ instead of $\mathbf{u}$ leads to a system where layer thickness, tracer constituents and PV are all transported by the same modified material derivative shown in (10). We refer to (8), (9), and (11) as the inviscid form of our eddy closure on potential vorticity. The extension of this closure to include diffusion along isopycnals is discussed after an analysis of the energetics of the inviscid, adiabatic system.

3. Analysis of energetics

While the above analysis indicates how to modify the momentum equation in order to transport PV with the same material derivative as the scalar fields, no mention is made of the energetics of the eddy closure.

a. The unmodified system

The isopycnal system analyzed in Section 2.a does conserve total energy. The kinetic energy equation is produced by adding $K*(1)$ and $\mu\mathbf{u}\cdot(2)$ to obtain

$$K\frac{\partial\mu}{\partial t}+\frac{\mu}{2}\frac{\partial\left|\mathbf{u}\right|^2}{\partial t}+\nabla\cdot(\mu\mathbf{u}K)=-\mu\mathbf{u}\cdot\nabla\phi. \tag{14}$$

In order to keep the nonlinear Coriolis force from generating spurious sources of kinetic energy, the kinetic energy *must* be obtained by taking the inner product of (2) with whatever velocity is used in the nonlinear Coriolis force. With $K=\frac{1}{2}\left|\mathbf{u}\right|^2$, equation (14) reduces to

$$\frac{\partial}{\partial t}(\mu K)+\nabla\cdot(\mu K\mathbf{u})=-\mu\mathbf{u}\cdot\nabla\phi. \tag{15}$$

The potential energy equation is obtained by taking $\phi*(1)$ to yield

$$\phi\frac{\partial\mu}{\partial t}=-\phi\nabla\cdot(\mu\mathbf{u}). \tag{16}$$

By adding (15) and (16), we obtain the total energy equation as

$$\frac{\partial}{\partial t}(\mu K)+\phi\frac{\partial\mu}{\partial t}+\nabla\cdot(\mu K\mathbf{u})+\nabla\cdot(\mu\phi\mathbf{u})=0. \tag{17}$$

It is important to note that the right-hand side (RHS) of (15) and (16) represent the conservative exchange of kinetic and potential energy due to the interaction between the pressure gradient force and the velocity field. This exchange is conservative only when the velocity used to transport layer thickness is the same velocity that is used to produce the kinetic energy equation.

If we integrate over the entire (x, y, ρ) domain with suitable boundary conditions on $\mathbf{u}$ and assume hydrostatic balance, then the total energy equation (17) becomes

$$\frac{\partial}{\partial t} \int_V \left[\mu K + \frac{g h^2}{2\rho_0} \right] dx\, dy\, d\rho = 0, \tag{18}$$

where g is gravity and ρ_0 is a reference density. (18) shows that $\mu K + gh^2/2\rho_0$ is a global invariant of the incompressible, Boussinesq and adiabatic system.

b. Energetics of the eddy closure for potential vorticity

We want the energy relations of the eddy closure to be well behaved, and mimic those of the unmodified system. In particular, we wish to keep the important physical property that the nonlinear Coriolis force does not contribute to the KE, which requires that the dot product of the momentum equation (11) is by the total velocity $\mathbf{U}$. Thus, the kinetic energy equation is formed by adding $K{*}(8)$ and $\mu\mathbf{U}{\cdot}(11)$ to obtain

$$K \frac{\partial \mu}{\partial t} + \mu \mathbf{U} \cdot \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mu \mathbf{U} K) = -\mu \mathbf{U} \cdot \nabla \phi. \tag{19}$$

While the kinetic energy has not yet been defined in the eddy closure, we note that, given suitable boundary conditions on $\mathbf{U}$, the term $\nabla \cdot (\mu \mathbf{U} K)$ vanishes when integrated over the entire domain; this result holds for any definition of K.

The potential energy equation is obtained by taking $\phi{*}(8)$ to produce

$$\phi \frac{\partial \mu}{\partial t} = -\phi \nabla \cdot (\mu \mathbf{U}). \tag{20}$$

The total energy equation is constructed by adding (19) and (20) to yield

$$K\frac{\partial \mu}{\partial t} + \mu \mathbf{U} \cdot \frac{\partial \mathbf{u}}{\partial t} + \phi\frac{\partial \mu}{\partial t} + \nabla \cdot (\mu \mathbf{U} K) + \nabla \cdot (\mu \mathbf{U} \phi) = 0. \tag{21}$$

Note that, as in the unmodified system, the terms on the RHS of (19) and (20) combine to produce a single term that vanishes when integrated over the entire domain.

The primary complication in deriving the energy relation for the eddy closure arises during the consideration of the definition of K. In the unmodified system, the first two terms in (14) combine when the obvious choice for K is made. In the eddy closure, the analogous terms *do not* combine because the transport velocity $\mathbf{U}$ differs, in general, from the mean velocity $\mathbf{u}$. If we focus on the first two terms in (21), we find that there is no definition of kinetic energy that makes these terms combine. However, if we choose

$$K = \frac{1}{2}(\mathbf{u} \cdot \mathbf{u}) + (\mathbf{u} \cdot \mathbf{u}^*) = \frac{1}{2}(\mathbf{u} \cdot \mathbf{U}) + \frac{1}{2}(\mathbf{u} \cdot \mathbf{u}^*), \tag{22}$$

this results in a total energy equation in the form

$$\frac{\partial}{\partial t}(\mu K) + \phi\frac{\partial \mu}{\partial t} + \nabla \cdot (\mu \mathbf{U} K) + \nabla \cdot (\mu \phi \mathbf{U}) = \mu \mathbf{u} \cdot \frac{\partial \mathbf{u}^*}{\partial t}. \tag{23}$$

Thus, the eddy closure has an energy relation analogous to the unmodified system that is not conservative to within the term shown on the RHS of (23). We made this choice for K because it has the usual first term from the mean velocity, and results in only a single RHS term in (23) that is small because it is proportional to $\partial \mathbf{u}^*/\partial t$.

4. Diffusion of PV along isopycnals

To this point we have restricted the analysis to inviscid dynamics. In addition to a Bolus transport velocity, the sub-grid effects on tracer mixing are also parameterized as a diffusion along isopyncals, as shown in Redi (1982). With an understanding of how to alter the momentum equation such that PV is transported by the total velocity, we analyze the impact on the momentum equation of assuming that PV is also diffused along isopyncals.

The hypothesis that PV is diffused along isopycnals leads to a PV equation of the form

$$\frac{D^{*}q}{Dt} = \frac{\nabla \cdot \left[\kappa \mu \nabla q\right]}{\mu}, \tag{24}$$

where we have used the small-slope approximation from Eq. 2 of Gent and McWilliams (1990). The premise of (24) is that PV is diffused along isopyncals in a manner exactly analogous to other tracers (Smith 1999).

Working backwards to determine the form of the momentum equation leading to (24) we obtain (11) with the additional term on the RHS of

$$\frac{\partial \mathbf{u}}{\partial t} + (\zeta + f)\, \mathbf{k} \times \mathbf{U} + \nabla \phi + \nabla K = \mathbf{k} \times \left[\kappa \mu \nabla q\right]. \tag{25}$$

If we repeat the PV derivation shown in Section 2.b based on (25), we obtain (24).

While the RHS of (25) results in PV being diffused along isopycnals in a manner exactly analogous to tracer diffusion, it is not obvious that the RHS of (25) satisfies other equally or more important constraints required of a diffusion-based closure modifying the momentum equation. In particular, momentum closures of this type should not damp velocity fields associated with solid body rotation (referred to as Condition I hereafter). In addition, momentum closures should result in a local, positive definite sink of kinetic energy (referred to as Condition II hereafter).

From a pragmatic perspective, we judge Conditions I and II as more essential properties of a momentum closure than ensuring that PV diffuses along isopyncals. While Conditions I and II might eclipse those related to PV dynamics, we would like to determine the extent to which all of these constraints can be satisfied.

The works of Wajsowicz (1993), Smith and McWilliams (2003), and Griffies (2004) derive a general form of diffusion-based momentum closures such that Condition I and II are satisfied. We find that the hypothesis that PV is diffused along isopycnals in a manner analogous to tracers, as stated in (25), is not entirely consistent with Conditions I and II.

The first problem with (25) is that the RHS can not be expressed as the divergence of a tensor. This deficiency is easily remedied by moving κ and μ inside the ∇ operator to obtain

$$\frac{\partial \mathbf{u}}{\partial t} + (\zeta + f)\,\mathbf{k} \times \mathbf{U} + \nabla\phi + \nabla K = \mathbf{k} \times [\nabla \kappa \eta]\,. \tag{26}$$

Moving μ inside the gradient implies that diffusion acts on absolute vorticity instead of potential vorticity. This idea is consistent with the finding from Haynes and McIntyre (1987) that diabatic processes can not alter the thickness-weighted PV and, as a result, absolute vorticity is conserved even under the diabatic rearrangement of isopycnal layers. While the RHS of (26) is the divergence of a stress tensor, the tensor is not skew-symmetric and, as a result, the stresses resulting from (26) damp solid body rotation.

In order to transform (26) such that Conditions I and II are satisfied, three modifications are required. First we assume that relative vorticity is diffused along isopycnals as opposed to the absolute vorticity, thereby altering the stress tensor to

$$\sigma = \begin{bmatrix} 0 & +\kappa\,\zeta \\ -\kappa\,\zeta & 0 \end{bmatrix} = \begin{bmatrix} 0 & \kappa\,(v_x - u_y) \\ \kappa\,(u_y - v_x) & 0 \end{bmatrix} \tag{27}$$

The stress tensor (27) is antisymmetric, which means it cannot satisfy Condition I. The simplest way to overcome this is to assume that the flow is in geostrophic balance and non-divergent so that $u_x + v_y = 0$. These two approximations, along with the choice of κ being a constant, lead to traditional Laplacian diffusion of the form

$$\frac{\partial \mathbf{u}}{\partial t} + (\zeta + f)\,\mathbf{k} \times \mathbf{U} + \nabla\phi + \nabla K = \kappa\nabla^2\mathbf{u}. \tag{28}$$

When κ varies in space, the appropriate form for the RHS of (28) that does not damp solid body rotation is

$$\frac{\partial \mathbf{u}}{\partial t} + (\zeta + f)\,\mathbf{k} \times \mathbf{U} + \nabla\phi + \nabla K = \nabla \cdot (\kappa\nabla\mathbf{u}) + J_{xy}\,(\kappa, \mathbf{k} \times \mathbf{u})\,, \tag{29}$$

see Wajsowicz (1993). Note that this requires a third approximation from (27) because, when κ varies, it leads to a Jacobian term with the opposite sign. We refer to (29) as the viscous form of our eddy closure for potential vorticity.

5. Discussion

We have postulated an eddy closure for potential vorticity that directly influences the evolution of momentum. We decompose the eddy closure into two parts, an inviscid modification of the momentum equation (11) followed by a viscous modification of the momentum equation (29). We believe that the viscous form of the eddy closure should be explored as a possible closure for use in ocean circulation models.

The inviscid form of the eddy closure is obtained by altering the manner in which the nonlinear Coriolis force in the momentum equation is computed (see (11)). This seemingly minor alteration allows us to derive an exact conservation relation for Ertel's potential vorticity within the framework of a Gent-McWilliams closure. The modification is to use the *transport* velocity instead of the *predicted* velocity in the computation of the nonlinear Coriolis force. This alteration leads to PV being conserved along particle trajectories defined by the transport velocity $\mathbf{U}$ that includes the influence of the Bolus velocity $\mathbf{u}^*$.

The inviscid form of the eddy closure significantly alters the energetics of the system relative to traditional GM closures. In traditional GM closures where layer thickness is transported by $\mathbf{U}$ and velocity is transported by $\mathbf{u}$, the conversion between resolved potential energy and resolved kinetic energy is not conservative. When the choice for the Bolus velocity is made following Gent and McWilliams (1990), the GM closure leads to a systematic global sink from resolved potential energy to unresolved scales. The intent of this global sink is to mimic the transfer of potential energy to unresolved, mesoscale kinetic energy. The energetics of the eddy closure developed above are significantly different. In this case the conversion between the resolved potential energy and resolved kinetic energy *is* conservative. While the end result of removing potential energy from the system might still be attainable through the dissipation of resolved kinetic energy, the energy pathway when using this eddy closure will be substantially altered relative to standard implementations of the GM closure. Understanding how this altered energy pathway impacts the global ocean circulation will be a critical aspect of the evaluation of this eddy closure.

While the inviscid form of the eddy closure produces an exactly conservative exchange of energy between the kinetic and potential energy reservoirs (see (21)), the closure does not lead to exact conservation of total energy. The reason for this discrepancy can be found in the kinetic energy equation of the eddy closure (19). The advective part of the material derivative is well-posed and the advecting velocity is the transport velocity $\mathbf{U}$. Yet we are unable to choose a kinetic energy to combine the time derivative terms due to the mixing of the predicted and transport velocities. As a result, the inviscid form of the eddy closure has a non-conservation in total energy that is proportional to the time derivative of the Bolus velocity (see RHS of (23)).

Previous studies that have explored the relationship between the GM closure and PV dynamics generally result in the addition of an undetermined gauge function appearing in the momentum equation (see, for example, Eq. 75 of Smith (1999)). The gauge function arises because, while an analysis of the PV dynamics can tightly constrain the rotational part of the velocity field, it offers no constraint on the divergent part of the velocity field. As a result, any purely divergent forcing can be added to the momentum equation and still obtain the same PV equation. In addition to being wholly unsatisfying, the unknown structure of the gauge function precludes the implementation of these closures in ocean circulation models. In contrast to previous studies, we have used the constraint of total energy conservation to remove the vagueness that accompanies the gauge function. If we add any non-zero gauge function to either the inviscid or viscous form of the eddy closure (shown in (11) and (29), respectively) then we "break" some of the key aspects of the energy analysis. As presented, both forms of the eddy closure result in the exact exchange of energy between its kinetic and potential forms. Adding any nonzero gauge function into the momentum equation would result in an inexact exchange of these forms of energy.

The inviscid form of the eddy closure on potential vorticity is surprisingly similar to the Lagrangian-Averaged Navier Stokes (LANS) closure, see (Holm 1999). In fact, our (11) has essentially the same functional form as the LANS closure when expressed in vector-invariant form (e.g. see Eq 1.4 in Gibbon and Holm (2006)). In the LANS closure, two velocities also exist; there is a transport velocity, termed the "smooth velocity" in the nomenclature of the LANS closure, and a predicted, mean velocity, termed the "rough velocity." In fact, the *only* differences between this eddy closure and the LANS closure is the definition of the potential on the RHS of the momentum equation (11), and the specification of the relationship between the transport velocity $\mathbf{U}$ and the mean velocity $\mathbf{u}$, i.e. the specification of $\mathbf{u}^*$. In GM-like closures, the $\mathbf{u}^*$ velocity is generally specified so as to produce a systematic flow of available mean potential energy into eddy kinetic energy. In the LANS closure, the specification of $\mathbf{u}^*$ is purely kinematic in nature being only a function of the mean velocity and a single specified parameter called α. Both closures lead to exact conservation of PV along trajectories defined by the transport velocity $\mathbf{U}$. The closures differ in terms of energetics; the LANS closure leads to exact conservation of total energy while our eddy closure for potential vorticity leads to non-conservation of total energy proportional to the time derivative of $\mathbf{u}^*$, as discussed immediately above. It would be informative to explore the implications of this striking resemblance between these two closures.

Following the analysis of PV dynamics in the inviscid system, we turn to the hypothesis that PV is diffused along isopycnals in a manner exactly analgous to other tracers. While one can readily find the form of dissipation on the RHS of the momentum equation that leads to PV being diffused along isopycnals, it has severe drawbacks compared to the standard form for horizontal viscosity used in ocean models in that it damps velocity fields associated with solid-body rotation and does not result in a local, positive-definite sink of kinetic energy. On the other hand, with three tenable assumptions, the closure form consistent with PV diffusion along isopyncals can be converted into a form that is significantly more comfortable. The assumptions required to transform the PV diffusion term into an acceptable form are that relative vorticity, not PV, is diffused along isopyncals and that to leading order the flow is in geostrophic balance. Furthermore, when the coefficient is variable in space then an additional approximation involving the reversal of the sign of the Jacobian term is required. With these assumptions, we are able to obtain a form of dissipation that is consistent with traditional Laplacian diffusion, that does not damp solid-body rotation and that locally dissipates kinetic energy. The result of this analysis is the viscous form of the eddy closure for potential vorticity shown in (29).

In our view, the most important result of the analysis of the viscous system is not related to the approximations required in the kinematic field in order to obtain a proper form of dissipation, but rather in the structure of the closure parameter. Our analysis clearly implies that the "horizontal viscosity coefficient" used in the momentum equation should be set equal to κ, which is the GM closure parameter. This had been hinted at in Gent and McWilliams (1996) in the quasigeostrophic limit. However, in all implementations of the GM scheme that we are aware of, the viscosity coefficient has always been chosen with different criteria and numerical values than the GM coefficient.

Any value that might be contained in the method proposed above can only be realized through the implementation of the closure in ocean circulation models. A recently developed numerical scheme has demonstrated the ability to mimic the important aspects of this eddy closure analysis within a discrete system (see Thuburn et al. (2009) and Ringler et al. (2009)). Specifically, the discretization of (11) leads to PV conserved along $\mathbf{U}$ trajectories and, along with the discretization of (8), leads to a conservative exchange between potential and kinetic energy, as shown in (23). The numerical method is applicable to virtually all types of meshes used in ocean climate models.

With the analytical analysis conducted above and a numerical scheme able to represent the relevant aspects this analysis, future work will concentrate on the implementation and evaluation of the eddy closure in ocean circulation models.

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